MEMORANDUM
RM-3291-PR
JUNE 1964

THE EFFECT OF
SURFACE-EVAPORATION KINETICS ON THE
SUBLIMATION INTO A BOUNDARY LAYER

A. F. Charwat

The RAND Corporation
1700 MAIN ST · SANTA MONICA · CALIFORNIA · 90406

PREFACE

The advent of slender high-speed re-entry bodies and particularly
slender lift-producing surfaces re-emphasizes the importance of phe-
nomena occurring near the leading edge. Previous work at The RAND
Corporation already considered such phenomena as the coupling of hy-
personic viscous induced pressure and mass addition at the surface --
typically a leading-edge problem showing measurable effects on the
aerodynamic force field. However, that analysis was based on an ide-
alized model of the flow, namely, on classical similarity solutions
for binary boundary layers.

This Memorandum explores the problem of the coupling between the
phase change of a sublimating surface and the flow near the leading
edge. The primary value of the results is the establishment of the
nature of this coupling and of its importance. The results contribute,
for example, to the decision regarding the conditions under which it
is realistic to proceed with studies of hypersonic induced pressure
interaction with surface blowing without simultaneously including the
coupling of the blowing rate with the flow field.

SUMMARY

The behavior of the laminar binary boundary layer with blowing is coupled with the kinetic-evaporation-rate law of the surface material. It is found that the solution of this combined problem exhibits an asymptotic behavior at large Reynolds number which is identified with the usual near-equilibrium solution in which the vapor near the wall is very nearly in thermodynamic phase-change equilibrium. However, the near-equilibrium solution is invalid in a region near the origin of the boundary layer, which is characterized by a length formed with physical parameters describing the basic flow and the surface properties. In this region, which is treated here approximately, the blowing parameter decreases to zero, the wall temperature increases, and the sublimation rate tends to an upper limit as the origin of the boundary layer is approached.

The principal result of the analysis is the estimation of the length of the region of transition to the near-equilibrium solution. This length is an independent characteristic of the problem and does not scale as do the boundary-layer properties (that is, with the Reynolds number). This implies that while the transitional length on typical re-entry vehicles is small and probably negligible unless the leading edge is very sharp, practical wind-tunnel test models can easily be affected over most of their chord.

CONTENTS

LIST OF SYMBOLS

A = numerical constant (Eq. (27))

a = speed of sound

B', B = blowing parameters

C = concentrations (mols/mol of mixture)

C_h = Stanton number, $q/\rho_e u_e (h_{aw} - h_w)$

C_o = viscosity-temperature law, $C_o = \rho_w \mu_w / \rho_e \mu_e$

c_p = specific heat

F = general functions, defined in the text where convenient for the discussion

H = dimensionless parameter, $\lambda M_s / RT_{aw_o}$

h = enthalpy

k = dimensionless parameter = $1 - T_T / T_{aw_o}$

M_i = molecular weight of component i (i = s for sublimating species, i = A for free-stream air)

$\mathcal{M}$ = free-stream Mach number

$\dot{m}_s$ = mass rate of sublimation

P = static pressure

Pr = Prandtl number

p = surface-vaporization (thermodynamic) constant, see Eq. (12)

q = heat-transfer rate

R = universal gas constant

T = temperature

T_{aw} = adiabatic wall-recovery temperature (T_w for $k\left(\dfrac{\partial T}{\partial y}\right)_w = 0$)

X = dimensionless parameter, see Eq. (33)

x = streamwise distance (origin at stagnation point)

α = constant defining the boundary-layer transfer properties (Eq. (16))

β = constant defining the recovery-temperature variation with blowing (Eq. (17))

Γ = the factor $\overline{\gamma} - 1/\overline{\gamma}$

γ = isentropic exponent of the gas

δ = leading-edge effect scale, Eq. (51) (units of length)

ϵ = vaporization coefficient (Eq. (10))

ζ = heat-transfer parameter, defined by Eq. (54)

λ = effective heat of sublimation

μ = coefficient of viscosity

ξ = heat-transfer parameter, for reference conditions without blowing (Eq. (20)) (units of pressure)

π = pressure-gradient parameter (similar solutions of boundary-layer equation)

ρ = density of the gas

σ = parameter defined in Eq. (36)

τ = dimensionless parameter (see Eq. 21))

Subscripts

e = free-stream conditions

o = reference conditions without blowing

T = stagnation conditions

w = wall conditions

∞ = conditions existing asymptotically for downstream

Subscript s pertains to sublimating species.

A bar over a symbol denotes average over binary mixture in the boundary layer.

I. INTRODUCTION

The complete solution of a flow field over a subliming surface represents an equilibrium among the rate of heat transfer to the wall, the rate of phase change of the surface material, and the rate of diffusion of the vapor evolved at the wall through the boundary layer. These conditions determine the surface temperature and the blowing rate.

The problem is usually treated as follows: The binary boundary-layer equations are solved with the wall temperature and the blowing rate treated as independent boundary conditions. The solution yields the temperature gradient, that is, the heat-transfer rate, as a parametric function of the wall temperature and the blowing rate. Since the blowing rate and the heat transfer are related by the effective heat of sublimation of the material, one more relation is needed to fix uniquely the two free parameters (wall temperature and blowing rate). At this point we make the approximation that the phase change occurs at thermodynamic equilibrium; that is, the wall temperature is the phase-equilibrium temperature at the existing partial pressure of the vapor near the wall (which is known from the solution of the binary boundary-layer equations). This relation suffices to complete the formulation.

The assumption of thermodynamic phase equilibrium at the surface is conceptually incorrect because at equilibrium the net mass transfer between phases is zero. Therefore, a more complete kinetic relation among the temperature, concentration, and rate of phase change is actually needed. This has been discussed in a number of papers,[1-5] but no solution including such a kinetic surface-evaporation condition has been obtained, nor have the implications of this phenomenon been fully explored.

The near-equilibrium solution is valid at sufficiently high Reynolds numbers, which can be illustrated by saying that when the Reynolds number is high, the impedance of the boundary layer to diffusion of vapor is very much higher than the impedance to surface

phase change, which is in series with it.[4] The latter is then negligible, and the coupling between the flow and the surface-evaporation phenomenon is diffusion limited. This statement leads to the question, What exactly are the lower limits of validity of this approximation?

If the surface-evaporation-rate law is considered, the results of the near-equilibrium approximation must be interpreted as saying that the difference between the actual wall temperature and the thermodynamic-equilibrium temperature of the vapor is small everywhere. The actual wall temperature is determined by the rate equation so as to supply the mass flux from the solid to the vapor phase. The equilibrium temperature is determined by the partial pressure which must exist near the wall to drive the mass flux across the boundary layer by pressure diffusion. Smallness is measured in comparison with the temperature difference driving the heat transfer; that is, the difference between adiabatic recovery and wall temperature.

The well-known result of near-equilibrium solutions for self-similar flows, for instance, is that the surface (equilibrium) temperature is constant, while the sublimation rate varies as the inverse square root of the Reynolds number. We may consider the Reynolds number as a unique measure of the distance from the origin of the boundary layer if the free stream is fixed and the wall temperature is constant But if the mass flux increases towards the origin of the boundary layer, then the wall temperature must increase according to the sublimation-rate law. The wall temperature cannot be constant, and therefore the Reynolds number cannot be a unique measure of distance. Also, the near-equilibrium approximation must break down at some distance towards the origin.

The properties of the boundary layer scale in terms of two parameters, the Reynolds number and the wall temperature (or more correctly, some ratio of free-stream temperature to wall temperature). Having recognized that the wall temperature is coupled to the problem through an independent rate equation, one concludes that the introduction of the kinetics of surface phase change brings in a new scale parameter. This scale has the form of a distance from the origin of the boundary layer for fixed free-stream conditions and given thermodynamic phase-change constants.

The condition for validity of the near-equilibrium approximation can now be stated as follows: The approximation is valid when the Reynolds number is much higher than the Reynolds number formed with the leading-edge scale described above. Moreover, the illustrative argument based on the relative magnitude of the impedance to mass transfer is seen to be incomplete; it does not consider the nonlinear coupling between the "impedance" and the driving potential. For instance, it implies that when the Reynolds number is very low the boundary-layer transfer impedance is low and the coupling between flow and sublimation is dominated by the surface impedance (rate limited). This is not true if the Reynolds number is low by virtue of low pressure at a given distance from the origin. It is only true if the Reynolds number is low by virtue of small distances from the leading edge.

The purpose of this study is to investigate the character of the rate-limited sublimation problem and, in particular, to determine the scale of this region. It is proper to make two remarks at this point. First, the analysis is based entirely on continuum boundary-layer concepts. This is equivalent to saying that the scale of the rate-limited region must be larger than some minimum distance required for validity of continuum boundary-layer concepts for the analysis to be meaningful. This situation turns out to be practically possible. Second, in regard to practical flows over somewhat blunted bodies, the statement "origin of the boundary layer" must be interpreted as meaning a virtual origin from which the boundary layer would start to attain a thickness and profile it has at the point under consideration. This implies that the distance to this virtual origin must be larger than the radius at the blunted nose.

The current trend towards finer re-entry shapes and the concern with problems such as the interaction between ablative mass addition and hypersonic viscous induced pressure[6]--typically a leading-edge phenomenon--tends to bring the transitional sublimation regime into the realm of practical problems

Finally, the present solutions for the rate-limited sublimation region involve a number of approximations in the treatment of the

boundary layer and should be interpreted mainly as a study of whether or not the problem is sufficiently significant to deserve a more rigorous and much more difficult analysis.

II. FORMULATION

The problem requires the simultaneous solution of both the boundary layer and surface phase-change rate equations. Phenomenologically, these fall into four groups describing, respectively, the energy and the mass-transfer properties of the boundary layer and the surface.

BOUNDARY-LAYER HEAT TRANSFER

The first is a solution of the classical binary boundary-layer equations which we consider to be uncoupled from the mass-diffusion equation,[7] implying a Lewis number approximately equal to one. We do not consider chemical reactions between the interdiffusing species. The solution has the form

$$\frac{C_h}{C_{h_o}} = 1 - F\left[B, \frac{M_s}{M_A}, \frac{dp_e}{dx}, \left(\frac{dT_w}{dx}\right)\right] \tag{1}$$

where B is a normalized blowing parameter

$$B = \frac{\dot{m}_s}{\rho_e u_e C_{h_o}} \tag{2}$$

The subscript o indicates reference conditions (nonablating surface), and the Stanton number C_h is defined on the basis of the adiabatic recovery enthalpy (temperature):

$$C_h = \frac{q}{\rho_e u_e (h_{aw} - h_w)} = \frac{q}{\rho_e u_e \bar{c}_p (T_{aw} - T_w)} \tag{3}$$

Equation (1) formally includes the influence of the pressure gradient, the temperature gradient, the variation of mean molecular weight, and gas properties through the boundary layer. This is reflected in the expression for the function F. Practically, only similarity solutions

are known, and for these the well-known linear approximation for F in terms of B holds to values of B on the order of 0.3. The numeric²¹ value of the proportionality constant and its dependence on Prandtl number, mass ratio, and pressure gradient (within the similar-flows family) has been discussed exhaustively in the literature.[1,7-10]

We propose to write an approximation to Eq. (1):

$$\frac{C_h}{C_{h_o}} = 1 - \frac{\alpha}{Pr} B$$

$$\alpha = \alpha\left(\frac{M_s}{M_A}, \frac{dp}{dx}, \frac{dT_w}{dx}\right)$$

(4)

Equation (4) is to be considered as a formal approximate expression of the behavior of the Stanton number for **B sufficiently small**, with α an unspecified function of the indicated parameters. The dependence on the Prandtl number is suggested by the similar solutions.[9]

It will be seen **a posteriori** that E goes from 0 when $Re_x = 0$ to an asymptotic value B_∞ when $Re_x \to \infty$. In the initial region the wall-temperature gradient is very strong and local similarity not valid; but Eq. (4) is then defendable as the first term of an expansion about B = 0. In the region of asymptotic approach, gradients are small[*] and, provided $B_\infty < 0.3$, Eq. (4) holds as a "local similarity" solution. It is difficult to imagine a situation in which essential errors in the trends exhibited in the transition region would result from the use of Eq. (4).

To the same degree of approximation the recovery factor for the binary boundary layer is expressed in terms of the recovery factor for the reference flow:

[*]The external pressure gradient is an independent parameter. It is assumed to be such that use of the "local-similarity" concept can be justified in regard to it.

$$r = \frac{T_{aw} - T_w}{T_{T_e} - T_w} = r_o - \frac{\beta}{Pr} B \tag{5}$$

The numerical values of $\beta = \beta(Pr, dp/dx, dT_w/dx, M_A/M_s)$ are less well defined from available solutions even for self-similar flows. It will be seen later, however, that the proportionality constants $\underline{\alpha}$ and $\underline{\beta}$ ap-pear in the solution only as a ratio. Thus, if they both vary in the same way with M_s/M_A; Pr, dp/dx, and dT_w/dx to a first approximation, the effect of this dependence is minimized.

THE HEAT BALANCE AT THE WALL

The thermal coupling between the flow field and the boundary is specified by

$$q = \dot{m}_s \lambda_s \tag{6}$$

where λ_s is the heat of sublimation of the wall material. This approx-imation neglects radiant and conductive heat losses. It also neglects the heat needed to bring the sublimating material from its initial temperature to sublimation temperature; both are fair approximations for low-temperature sublimators.

If we define

$$B' = \frac{\dot{m}_s}{\rho_e u_e C_h} = \frac{(h_{aw} - h_w)}{\lambda_s} \tag{7}$$

and use the identity

$$\frac{B}{B'} = \frac{C_h}{C_{h_o}} = \frac{T_{aw_o} - T_w}{(T_{aw} - T_w)} \tag{8}$$

we obtain the second well-known relation of simple theories.

BOUNDARY-LAYER MASS-TRANSFER CHARACTERISTIC

The solution of the boundary-layer diffusion equation for the transfer of the sublimated material away from the wall is[7]

$$- \frac{C_s - C_{sw}}{C_{sw}} = \frac{(h_T - h_w)}{(h_{Te} - h_w)} \tag{9}$$

This form is strictly true only for the case where both Le and Pr are equal to one; in this case the diffusion equation and the equation for the distribution of total enthalpy in the boundary layer are identical and concentration and energy profiles are similar. The effect of Pr on the similarity of the profiles is minor,[7,10] weaker than its effect on C_h itself. This justifies the use of Eq. (8) without also setting Pr = 1 in Eq. (1) and equations derived from it.

Combining Eq. (9) with Fick's law (see Ref. 7), we obtain an expression for the wall concentration as a function of the rate of sublimation at the wall:

$$\dot{m}_s \left(1 - C_{sw} \right) = C_{sw} \frac{q}{h_{Te} - h_w} = C_{sw} \rho_e u_e \frac{h_{aw} - h_w}{h_{Te} - h_w} C_h \tag{10}$$

PHASE-CHANGE KINETICS

A last equation couples the concentration of the sublimated species in the boundary layer to the sublimation phenomenon itself. The net rate of exchange of surface material across the surface-potential barrier is

$$\dot{m}_s = \epsilon \sqrt{\frac{M_s}{2\pi RT_w}} (P_{s\,equ} - P_{s_w}) = \epsilon \frac{P\bar{M}}{\sqrt{\pi R M_s T_w}} (C_{s\,equ} - C_{sw}) \tag{11}$$

where ϵ is an empirical "vaporization coefficient," $P_{s\,equ}$ and $C_{s\,equ}$ are the equilibrium partial pressure and concentration corresponding to the wall temperature, and P_{s_w} and C_{s_w} are the actual partial pressure

-9-

and concentration of the subliming material immediately over the surface in the boundary layer. For a two-component mixture one has the following auxiliary relations between the concentrations and the molecular weights ($\bar{M}$ is the mean molecular weight of the mixture):

$$\frac{P_s}{P_e} = \frac{\bar{M}}{M_s} C_s = \left\{ 1 + \left[\frac{1}{C_s} - 1 \right] \frac{M_s}{M_A} \right\}^{-1}$$

(12)

$$\bar{M} = \left[\frac{C_s}{M_s} + \frac{1}{M_A} - \frac{C_s}{M_A} \right]^{-1}$$

$$C_A + C_s = 1$$

An important characteristic of the phenomenon is that there is a maximum rate of escape of surface atoms, which occurs when the concentration of the material in the surrounding gas is zero and which depends only on the wall temperature.[2] Several analytical expressions for the value of $P_{s\ equ}$ that determines this maximum can be written down, depending on the subtlety of the microscopic model. The simplest one, corresponding to the integral of the Clausius-Clapeyron equation, is

$$P_{s\ equ} = p \exp \left(- \frac{M_s \lambda_s}{RT_w} \right)$$

(13)

where p is a constant. This yields for the maximum (forward) vaporization rate the expression

$$\dot{m}_{s_v} = \epsilon p \sqrt{\frac{M_s}{2\pi RT_w}} \exp \left(- \frac{M_s \lambda_s}{RT_w} \right)$$

(14)

The set of Eqs. (1), (3), (9), and (13), together with an auxiliary caloric equation of state defining $\bar{c}_p$

$$\bar{c}_p = \frac{h_{aw} - h_w}{T_{aw} - T_w} \tag{15}$$

complete the general definition of the problem. We shall assume in this analysis that $\bar{c}_p$ can be treated as an <u>a priori</u> (or iteratively) determined constant, not dependent on the concentrations.[11]

The following dimensionless parameters, which depend only on the properties of the surface and/or the free-stream conditions, are now defined:

$$H = \frac{\lambda_s M_s}{RT_{aw_o}} \tag{16}$$

$$\Gamma = \frac{R}{M_A \bar{c}_p} = 1 - \frac{\bar{c}_v}{\bar{c}_p} = \frac{\bar{\gamma} - 1}{\bar{\gamma}} \tag{17}$$

$$\tau = -\frac{T_T - T_e}{T_{aw_o}} \frac{\beta}{Pr} \tag{18}$$

$$k = 1 - \frac{T_T}{T_{aw_o}} = \frac{\frac{\gamma - 1}{2} M^2 (r_o - 1)}{1 + \frac{\gamma - 1}{2} M^2 r_o} \tag{19}$$

$$\zeta = \frac{1}{e} \sqrt{2\pi \frac{R}{M_s} T_{aw_o}} \, \rho_e u_e C_{h_o} \sqrt{Re_x} \tag{20}$$

The function ζ represents the product $C_{h_o} \sqrt{Re_x}$ which, at least for self-similar flows, is a constant. Otherwise this product varies with x through the streamwise pressure and the wall-temperature

gradients. While questions must be raised as to its behavior near the singular point x = 0, which are mentioned in Section IV, at large Reynolds numbers it is certainly a finite, slowly varying quantity.

Eliminating $\dot{m}_s$ between Eqs. (2) and (11) and rearranging, we write

$$\frac{\rho_e u_e C_{h_o}}{\epsilon} \sqrt{\frac{2\pi R T_{aw_o}}{M_S}} \, B = \sqrt{\frac{T_{aw_o}}{T_w}} \left[P_{s\,equ} - P_{s_w} \right] \tag{21}$$

By straight substitutions we transform it into an equation in only one unknown, which is B:

$$\frac{B\zeta}{\sqrt{Re_x}} = \left[\frac{1 - \frac{\alpha}{Pr} B}{1 + B\left(\tau - \frac{M_A}{M_s} \Gamma H - \frac{\alpha}{Pr}\right) - \frac{\alpha\tau}{Pr} B^2} \right]^{1/2}$$

$$\left[pe^{-H} \exp\left(H \frac{B\left(\tau - \frac{M_A}{M_s} \Gamma H\right) - \frac{\alpha\tau}{Pr} B^2}{1 + B\left(\tau - \frac{M_A}{M_s} \Gamma H - \frac{\alpha}{Pr}\right) - \frac{\alpha\tau}{Pr} B^2} \right) \right. \tag{22}$$

$$\left. - P_e \left(\frac{k + B\left(\tau - \frac{M_A}{M_s} \Gamma H - k\frac{\alpha}{Pr}\right) - \frac{\alpha\tau}{Pr} B^2}{k + B\left(\tau - \frac{M_A}{M_s} \Gamma H - k\frac{\alpha}{Pr}\right) - \frac{\alpha\tau}{Pr} B^2 + \Gamma H\left(\frac{\alpha}{Pr} B - 1\right)} \right) \right]$$

For clarity, a few intermediate steps in this transformation are given in the following. The left-hand sides of Eqs. (21) and (22) involve simply the definition of ζ, Eq. (20). The first factor on the right-hand side follows from Eqs. (8) and (4), which yield

$$B' = \frac{B}{1 - \frac{\alpha}{Pr} B} \tag{23}$$

Substituting for B' from Eq. (7) one obtains

$$
B = \frac{-\left(\dfrac{T_w}{T_{aw}} - 1\right)}{\dfrac{\lambda}{\bar{c}_p T_{aw}} - \dfrac{\alpha}{Pr}\left(\dfrac{T_w}{T_{aw}} - 1\right)}
\tag{24}
$$

From Eqs. (5) and (18) one has

$$
T_{aw} = T_{aw_o} (1 + \tau B)
\tag{25}
$$

which, introduced into Eq. (24) and solved for the temperature ratio, gives

$$
\frac{T_w}{T_{aw_o}} = 1 + \frac{B\left(\tau - \Gamma H \dfrac{M_A}{M_s} - \dfrac{\alpha \tau B^2}{Pr}\right)}{1 - \dfrac{\alpha}{Pr} B}
\tag{26}
$$

The first term in brackets in Eq. (22) is simply Eq. (13) written in the form (see Eqs. (16) to (19))

$$
P_{s\ equ} = pe^{-H_e H} \frac{\dfrac{T_w}{T_{aw_o}} - 1}{\dfrac{T_w}{T_{aw_o}}}
\tag{27}
$$

with the temperature ratio as given by Eq. (26).

The second term in brackets follows from Eqs. (12) and (10). From Eq. (10) we derive the following expression for C_{sv} in terms of B:

$$C_{sw} = \frac{B\left(\tau - \frac{M_A}{M_s} \Gamma H - k \frac{\alpha}{Pr}\right) + k - \frac{\alpha\tau}{Pr} B^2}{B\left[\tau - \frac{M_A}{M_s} \Gamma H - \frac{\alpha}{Pr}\left(k - \frac{M_A}{M_s} \Gamma H\right)\right] + k - \frac{M_A}{M_s} \Gamma H - \frac{\alpha\tau}{Pr} B^2} \qquad (28)$$

This equation, after substitution in Eq. (12), leads to an expression for P_{sw} which is exactly the second term in brackets of Eq. (22).

III. ASYMPTOTIC BEHAVIOR DOWNSTREAM

For large values of Re the left-hand side of Eq. (22) vanishes and the equation yields a non-zero[*] value $B = B_\infty$ given by the solution of the factor in brackets on the right-hand side.

Note that this corresponds exactly to stating

$$\left[P_{s\ equ} - P_{sw} \right] = 0$$

The solution is identified with the "equilibrium solution" in which the partial pressure of the vapor, and consequently also its concentration and temperature, are values corresponding to thermodynamic phase-change equilibrium.

In the present formulation the solution is given in terms of the constant p in the analytical expression for $P_{s\ equ}$ (instead of specifying $T_w = T_{s\ equ}$ separately from thermodynamic tables). A convenient graphical procedure is obtained defining

$$X_\infty = \frac{T_w}{T_{aw_o}} - 1 = \frac{B_\infty\left(\tau - \Gamma H \frac{M_A}{M_s}\right) - B_\infty^2 \frac{\alpha}{Pr} \tau}{1 - \frac{\alpha}{Pr} B_\infty} \tag{29}$$

in terms of which the asymptotic solution takes the form

$$\ln \frac{P_e}{p} = - \frac{H}{X_\infty + 1} + \ln \left[\frac{X_\infty + k - \Gamma H}{X_\infty + k} \right] \tag{30}$$

This is plotted in Fig. 1 for a particular value of Γ. For any external static pressure P_e and parameters describing the properties of the sublimating material (p and H), one obtains a unique value for the wall-temperature function X_∞. With this value of X_∞ and further

_____

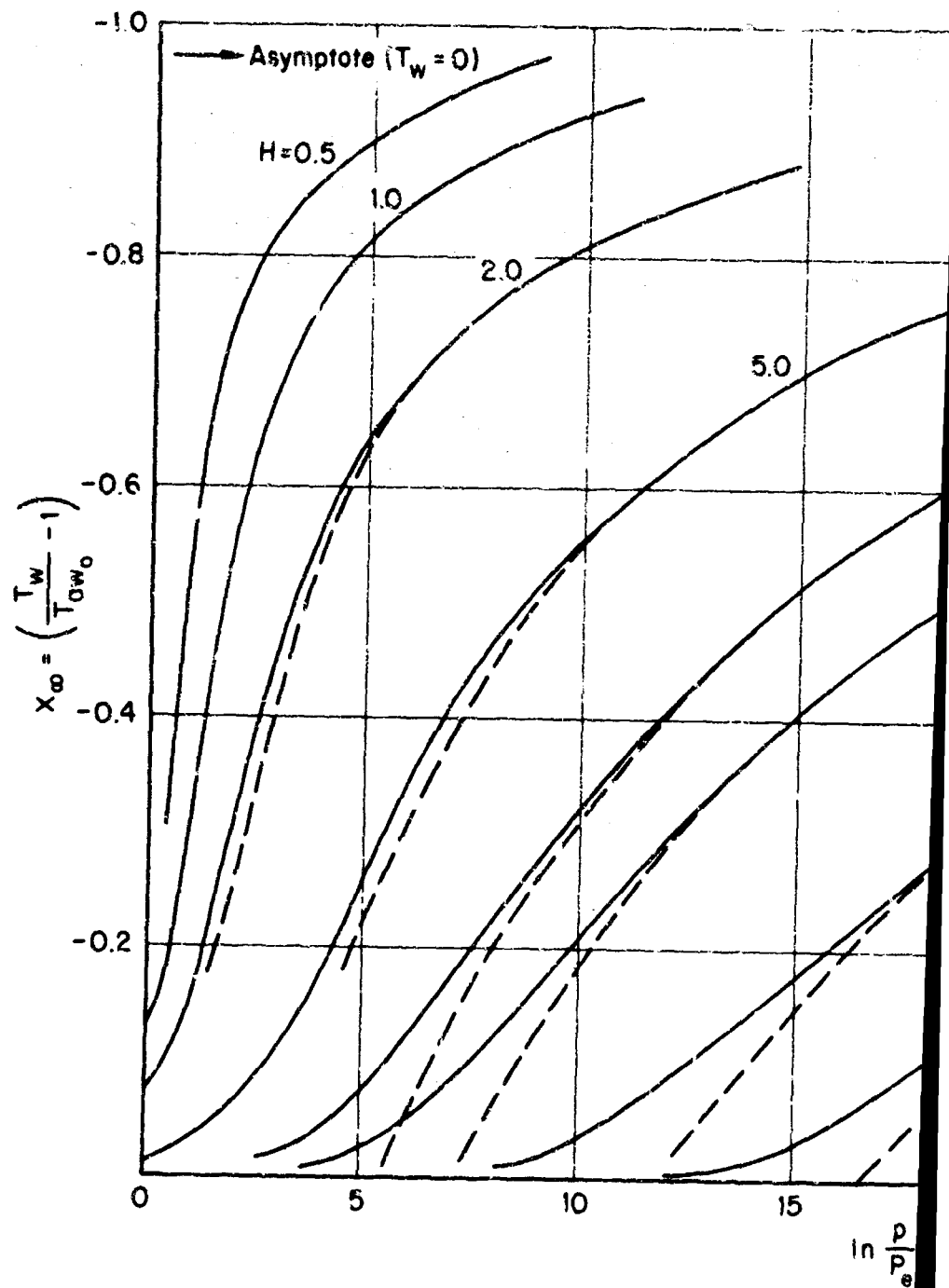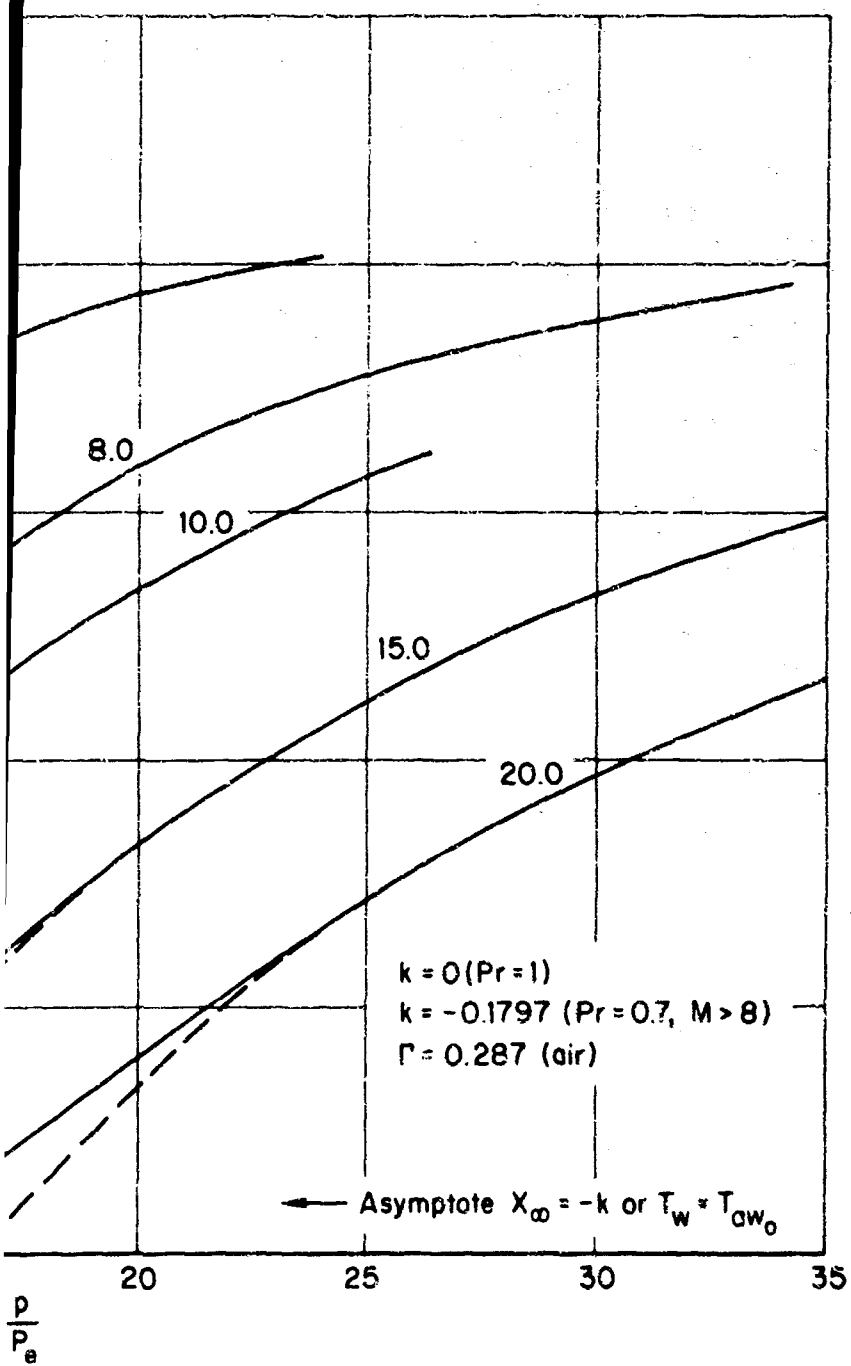[*] The singularity B = 0 is uninteresting. It corresponds simply to flow without sublimation.

Fig. 1 — Wall - tempe

The graph shows curves labeled 8.0, 10.0, 15.0, and 20.0 with the horizontal axis $\frac{p}{P_e}$ ranging from 20 to 35.

$k = 0 \, (Pr = 1)$

$k = -0.1797 \, (Pr = 0.7, \, M > 8)$

$\Gamma = 0.287 \, (air)$

$\longleftarrow$ Asymptote $X_\infty = -k$ or $T_w = T_{aw_o}$

perature function

parameters pertaining to the reference flow and the sublimating material (α/Pr_T), Fig. 2, which is a plot of Eq. (29), yields the value of the blowing parameter B_∞.

Figure 3 is an auxiliary figure giving the variation of α/Pr_T. This can be written as

$$\frac{\alpha}{Pr_T} = \frac{\dfrac{\alpha}{\beta}}{\left(1 - k\right)\left(1 - \dfrac{T_e}{T_T}\right)} \tag{31}$$

The two proportionality constants α and β (see Eqs. (4) and (5)) appear as a ratio. Therefore, their dependence on the principal variables of the binary boundary-layer problem, which are M_s/M_A, Pr, and pressure and wall-temperature gradient, is minimized. If one makes the hypothesis that both vary in the same fashion with these parameters, which seems probable, then the combination α/Pr_T is to a very good approximation a function only of the free stream.
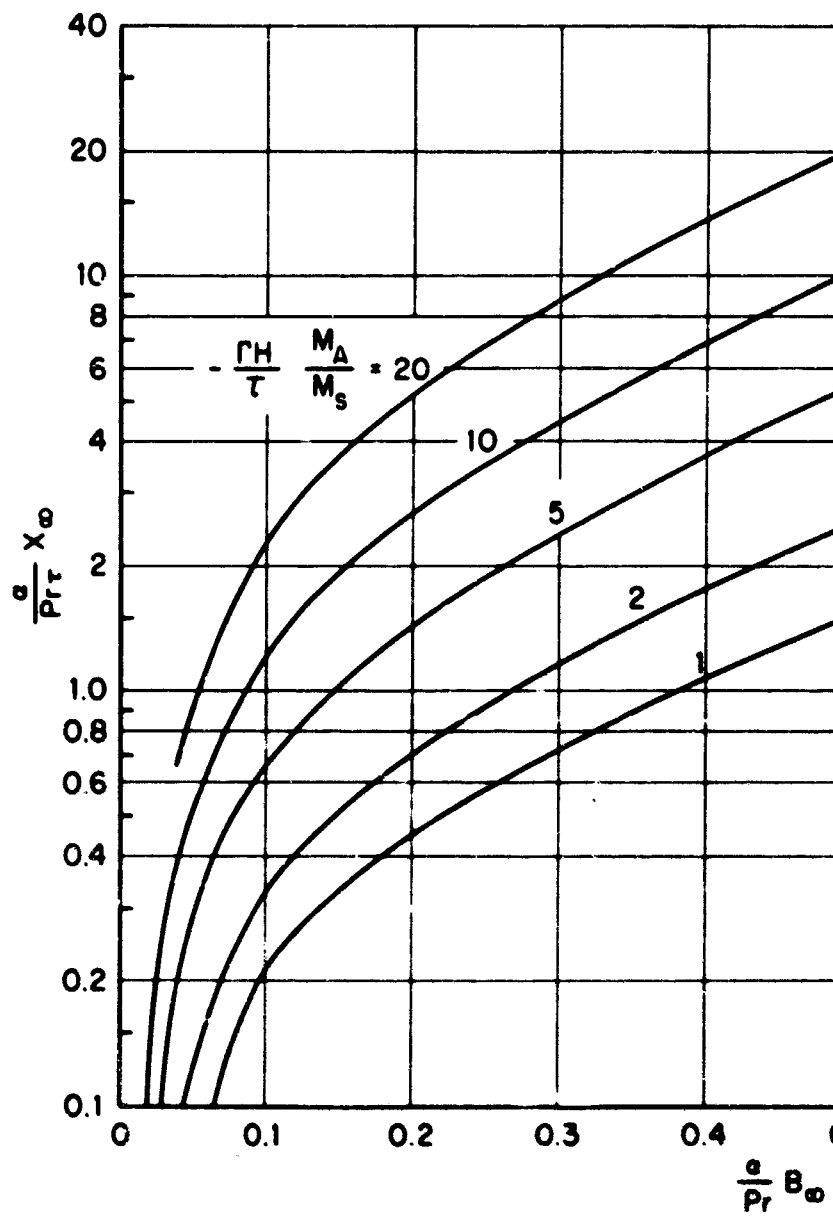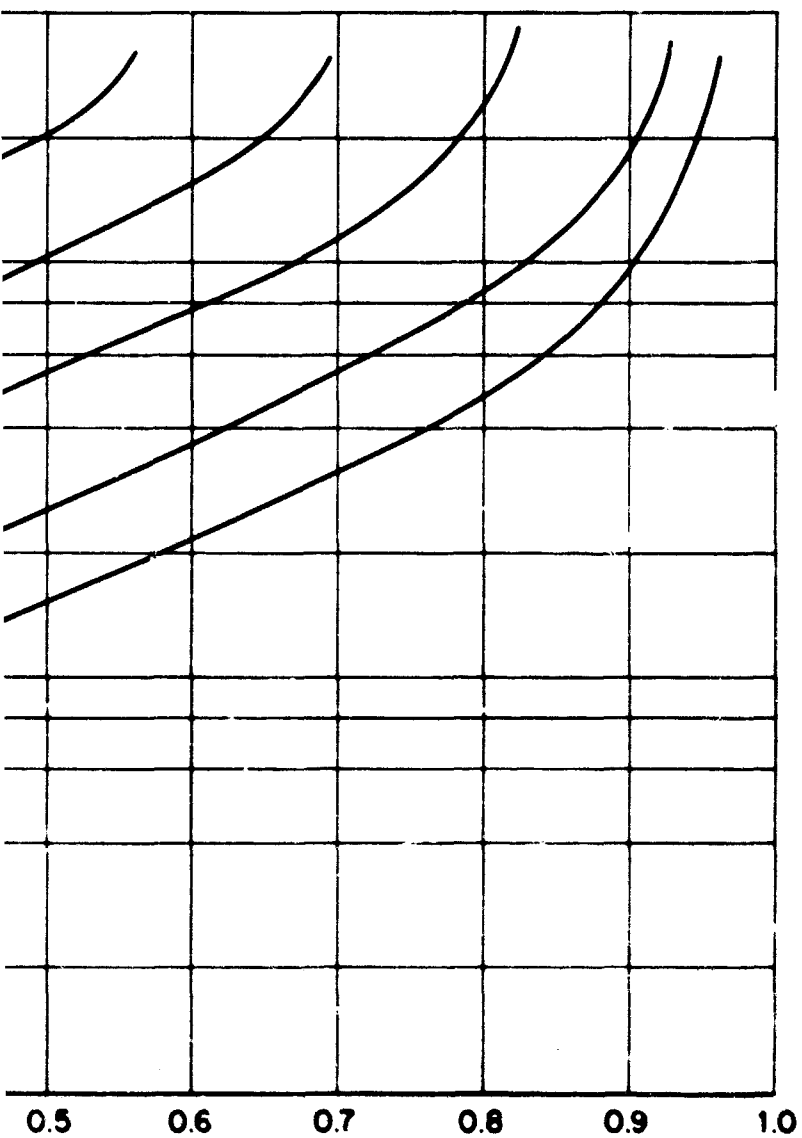
Fig. 2 — Asymptotic blowing parameter

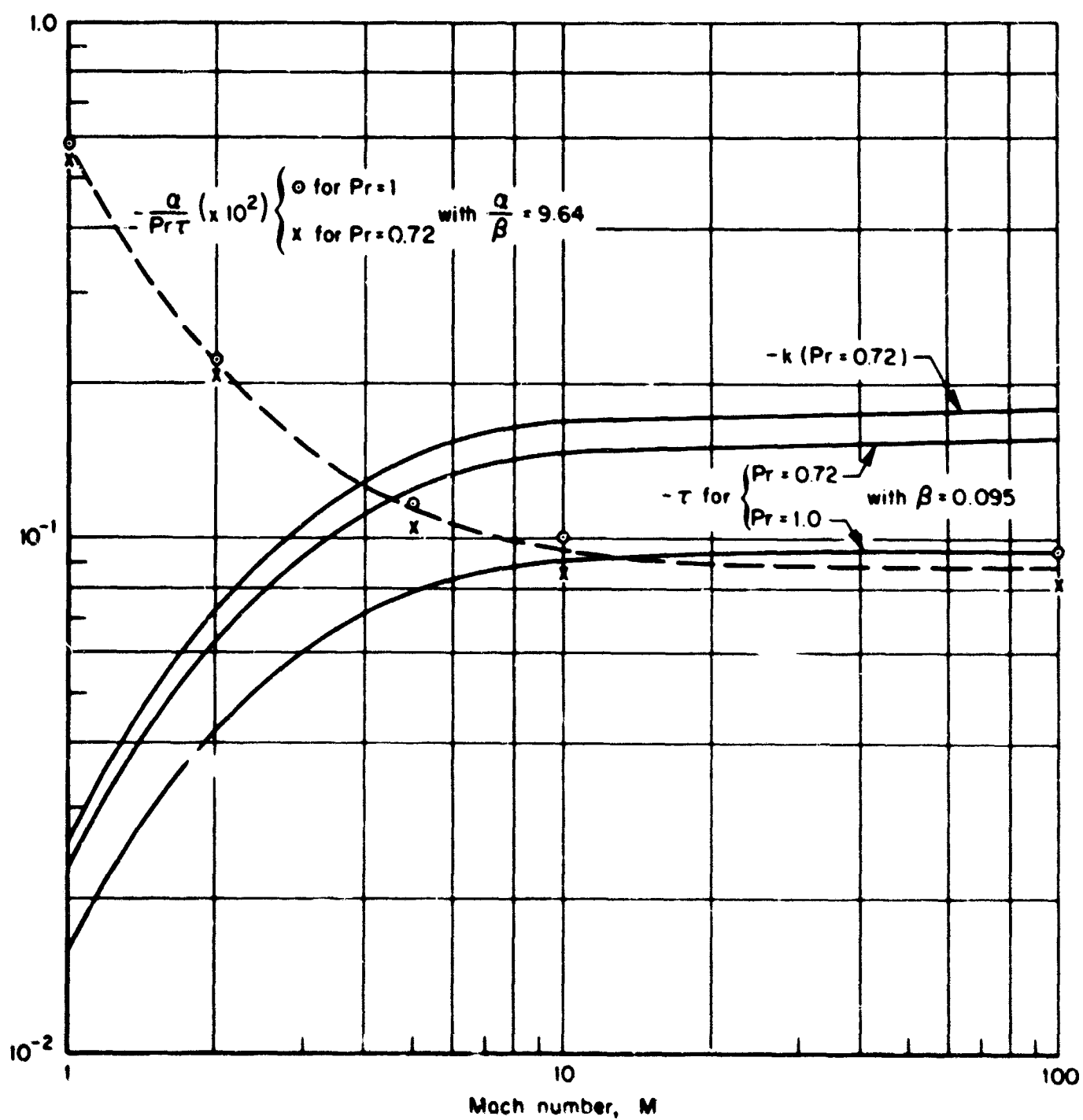|					
0.5	0.6	0.7	0.8	0.9	1.0

er versus wall - temperature function

Fig. 3 — Variation of some Mach - number - dependent parameters ($\pi = 0$)

IV. BEHAVIOR NEAR THE STAGNATION POINT

Equation (31) has the form

$$\frac{\xi}{\sqrt{Re_x}} = \frac{f(B)}{B} \tag{32}$$

The parameter ξ appearing on the left-hand side of Eq. (32) represents the product $C_{h_0}\sqrt{Re_x}$. The assumption is made that this product is a constant, as it is for self-similar boundary-layer solutions. Now, since $\dot{m}_s = \lambda q$ is bounded according to the surface-evaporation equation, the assumption implies that the wall temperature tends to the recovery temperature towards the leading edge ($Re_x \to 0$). It follows that the wall temperature is variable, and $C_{h_0}\sqrt{Re_x}$ = constant can only be good in the sense of "local similarity."

It is worth noting that the above implies two statements of unequal importance to the present analysis. The more important one concerns the <u>behavior</u> of C_{h_0}; that is, that C_{h_0} grows without bounds towards the leading edge. This leads to the result that $B \sim \dot{m}_s/C_{h_0}$ tends to zero there and, thus, that B varies between zero and B_∞ over the entire region of interest. The second and less important one concerns the numerical accuracy of the assumption $C_{h_0}\sqrt{Re_x}$ = constant. It is undoubtedly poor very near the origin where the wall-temperature gradients are large, but it is probably satisfactory in the region of asymptotic approach to the near-equilibrium solution downstream of the leading edge.

It is not possible to discuss conclusively the difficult problem of the singularity at the leading edge. It must be accepted on the basis of heuristic arguments and the analogy with the behavior of ordinary boundary layers at the leading edge, which involves similar difficulties. Physically, the behavior outlined in what preceded is quite reasonable. Moving upstream towards the leading edge, the heat flux to the wall increases, and the wall temperature must rise to permit an increased rate of sublimation.

Returning to Eq. (32), the right-hand side is expressible in the form of a series in the interval $0 < B < B_\infty$. If the asymptotic blowing parameter B_∞ is reasonably small, which is already implied by the use of the linear-blockage equation, Eq. (4), a limited number of terms of the series will represent the behavior of the function throughout the range.

The series is

$$\frac{f/f(0)}{\sqrt{Re_x}} = \frac{1}{B} + \frac{f'(0)}{f(0)} + \frac{1}{2} B \frac{f'(0)}{f(0)} + \ldots \tag{33}$$

with the following expressions for the function and its derivative evaluated at the origin:

$$f(0) = pe^{-H} - \frac{kP_e}{k - \Gamma H} \tag{34}$$

$$\frac{f'(0)}{f(0)} = \left(\tau - \Gamma H \frac{M_A}{M_s}\right)\left[\frac{3}{2} + \sigma \frac{P_e}{p} \frac{e^H}{\Gamma H}\right] \tag{35}$$

The parameter σ has the form

$$\sigma = \frac{(\Gamma H)^2 pe^{-H} + k(k - \Gamma H)\Gamma Hpe^{-H}}{(\Gamma H)^2 pe^{-H} + k(k - 2\Gamma H) pe^{-H} + kP_e(\Gamma H - k)} \tag{36}$$

and it was defined so as to become one when $k = 0$ (Prandtl number unity).

It is interesting to record the initial behavior of the pertinent physical parameters of the problem. First-order expansions are given below:

$$C_{sw} = \frac{k}{k - \Gamma H \frac{M_A}{M_s}} \left[1 - \frac{\Gamma H \frac{M_A}{M_s} \left(\tau - \Gamma H \frac{M_A}{M_s} \right)}{k \left(k - \Gamma H \frac{M_A}{M_s} \right)} B + \ldots \right] \tag{37}$$

$$P_{sw} = \frac{P_e k}{k - \Gamma H} \left[1 - \frac{\left(\tau - \Gamma H \frac{M_A}{M_s} \right) \Gamma H}{k(k - \Gamma H)} B + \ldots \right] \tag{38}$$

$$P_{s\,equ} = pe^{-H} \left[1 + H \left(\tau - \Gamma H \frac{M_A}{M_s} \right) B \ldots \right] \tag{39}$$

$$\frac{T_w}{T_{aw_o}} = 1 - \left(\tau - \Gamma H \frac{M_A}{M_s} \right) B + \ldots \tag{40}$$

$$\dot{m}_s = \frac{\epsilon}{\sqrt{2\pi \frac{R}{M_s} T_{aw_o}}} \frac{\epsilon}{\sqrt{Re_x}} B + \ldots \tag{41}$$

V. SCALE OF THE LEADING-EDGE EFFECT

The first two terms of the series expansion, Eq. (33), vary as $Re_x \to \infty$ to the asymptotic limit of $B = f(0)/f'(0)$. This limit is not correct because it does not equal B_∞ unless B_∞ tends to zero. However, we know B_∞ independently from the full solution at the asymptotic limit, Eqs. (29) and (30), or Figs. 1 and 2.

The argument suggests that if one replaces in the two-term expansion Eq. (33)

$$- \frac{f'(0)}{f(0)} \to \frac{1}{B_\infty} \tag{42}$$

one obtains a good approximation to the behavior of B for all Re_x and arbitrary values of B_x, provided B_x is sufficiently small to justify the linear expression for the blockage factor, Eq. (4).

Accordingly, an approximate equation for B is as follows:

$$\frac{\xi/f(0)}{\sqrt{Re_x}} = \frac{1}{B} = \frac{1}{B_x} \tag{43a}$$

$$\frac{B}{P_x} = \frac{\sqrt{Re_x}}{\sqrt{Re_x} + \sqrt{\Delta}} \tag{43b}$$

where Δ is a dimensionless scale factor

$$\Delta = \left[\frac{\Delta B_x}{f(0)} \right]^2 \tag{44}$$

The corresponding approximation for the variation of mass-addition rate is, from Eq. (45)

$$\dot{m}_s = \frac{\dot{c}}{\sqrt{2 - \frac{R}{M_s} T_{aw_o}}} \frac{\dot{B}_x}{\sqrt{Re_x} + \sqrt{\ldots}}$$

$$\tag{45}$$

$$= \frac{\epsilon f(0)}{\sqrt{2 - \frac{R}{M_s} T_{aw_o}}} \frac{\sqrt{\ell}}{\sqrt{Re_x} + \sqrt{\ell}}$$

For pointed bodies the trends under study are demonstrated more emphatically if one defines a length ℓ by

$$\ell = \frac{\rho_e u_e \ell}{\mu_e} \tag{46}$$

which is explicitly given as a function of reference-flow and sublimating-material parameters by[*]

$$\ell = B_x^2 \left(e^{-H} - \frac{kP_e}{k - 1 H} \right)^{-2} \frac{2 - }{c^2} \rho_e u_e \mu_e \frac{R}{M_s} T_{aw_o} \left(C_{h_o} \sqrt{Re_x} \right)^2 \tag{47}$$

Equations (43) and (45) can then be written

$$\frac{B}{B} = \frac{\sqrt{x/\ell}}{\sqrt{x/\ell} + 1} \tag{48}$$

$$\dot{m}_s = \frac{\epsilon f(0)}{\sqrt{2 - \frac{R}{M_s} T_{aw_o}}} \frac{1}{\sqrt{\frac{x}{\ell}} + 1} \tag{49}$$

This equation shows how, for flow from a pointed leading edge, $\dot{m}_s$ starts from its maximum value dictated by kinetic considerations and decreases to its asymptotic variation downstream, which is proportional

_____

[*]Note that $C_{h_o} \sqrt{Re_x}$ is a constant with x.

to the inverse square root of the streamwise distance.

Similar explicit expressions for the variation of the other parameters are obtained by substituting B from Eq. (48) into the first-term expansions, Eqs. (37) through (40). We see that the concentration and the partial pressure of the sublimating species reach constant values at the origin, the magnitude of which depends only on k (the Prandtl number) and which are zero for the particular case of Pr = 1. The wall temperature tends to the adiabatic recovery temperature for zero blowing at the origin; that is, to the stagnation temperature if Pr = 1. The blowing parameter itself tends to zero at the origin of the boundary layer.

The first-term expression, Eq. (48), is plotted in comparison with a numerical solution of the full Eq. (22) in Fig. 4 for a particular set of physical parameters.

It is of interest to consider further the parameter Δ, which determines the scale of the adjustment to the asymptotic solution. Figure 5 shows the variation of this quantity in the form (see Eqs. (44) and (34))

$$\frac{\sqrt{\Delta P_e}}{\xi} = \frac{B_\infty}{\frac{p}{P_e} \exp(-H) \cdot \frac{k}{k - \Gamma H}} \tag{50}$$

The principal independent parameters in Eq. (50) are p/P_e and H (note k = 0 for Pr = 1). These also determine B_∞ uniquely (see Figs. 1 and 2), provided that the secondary parameters $\alpha/Pr\tau$ and τ can be considered constant, which is indeed very closely true above a Mach number of 8 (see Fig. 3).

Equation (50) is plotted in Fig. 5, and values of B_∞ are also shown to bring out the physical problem. The domain of practical interest is at high values of p/P_e and relatively large values of B_∞. The maximum of these curves, which occurs when B_∞ tends to zero, is not interesting in itself, but it serves to demonstrate the influence of another parameter of importance which is the mass ratio M_s/M_A. Indeed, in this region a simplified explicit solution can be written.
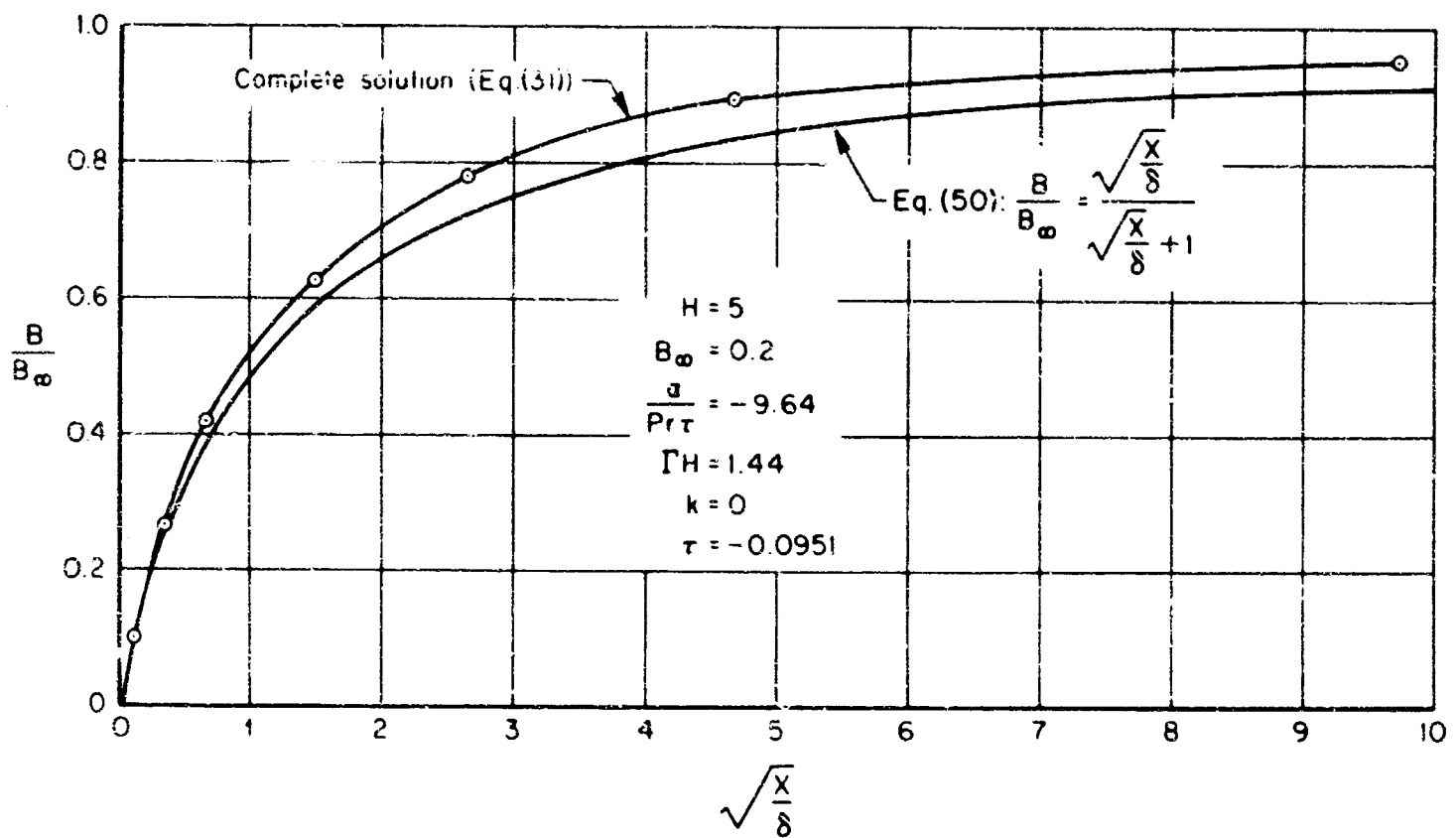
Fig. 4 — Comparison of full solution with the approximate
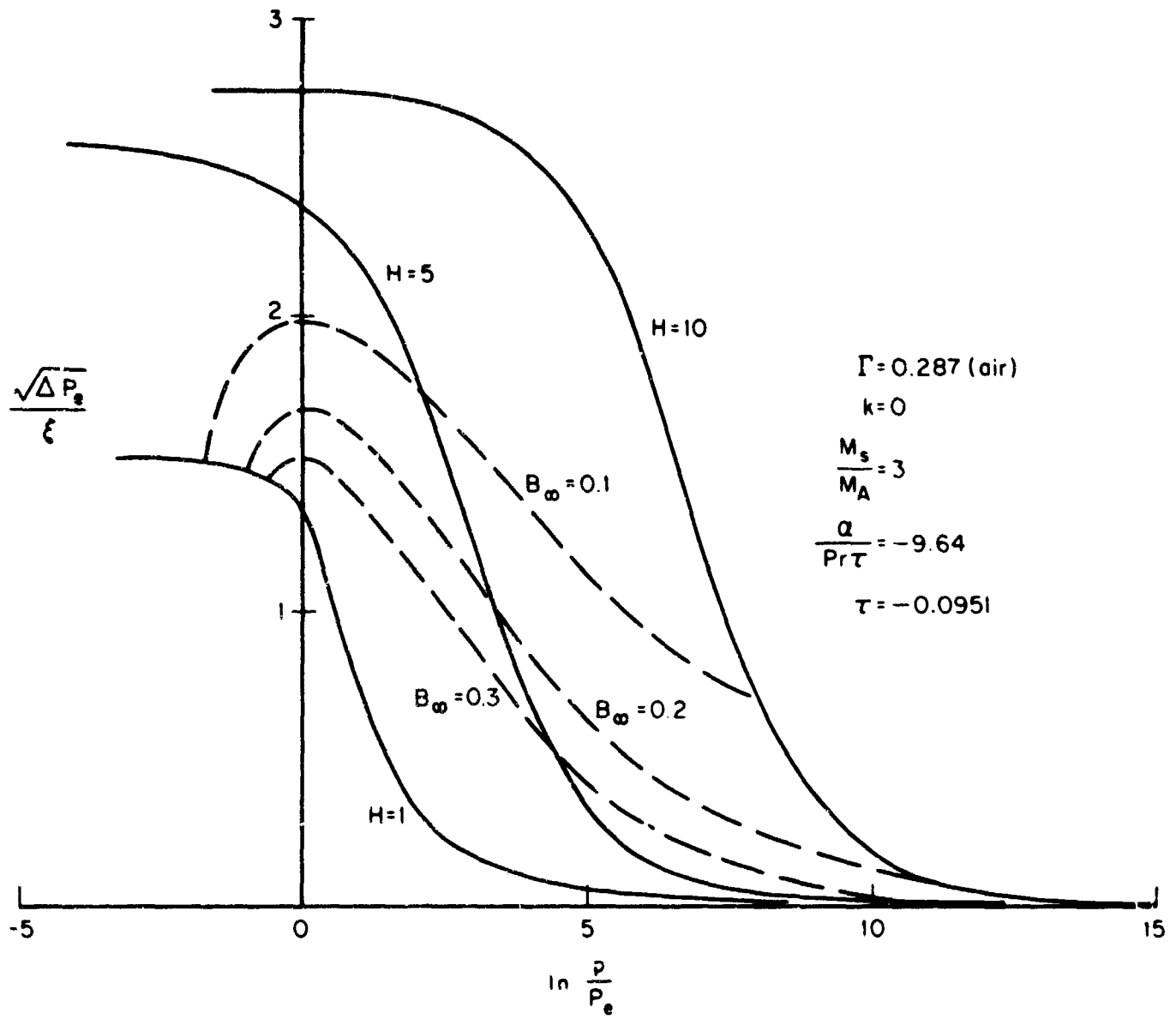modified first-term expansion

Fig. 5 — Typical variation of the leading-euge scale parameter, showing lines of B_∞ = constant

Low values of p/P_e correspond to small values of X_∞ (see Eq. (30)). Since k is normally near zero, the approximate form of Eq. (30) is

$$-(X_\infty + k) = \frac{(\Gamma H - k)pe^{-H}}{P_e} \tag{51}$$

Moreover, when X_∞ tends to zero, B_∞ tends to zero also. One can neglect the quadratic term in Eq. (29) which becomes

$$\frac{\alpha}{Pr} B_\infty = \frac{\frac{\alpha}{Pr\tau} X_\infty}{1 - \frac{\Gamma H}{\tau} \frac{M_A}{M_s}} \tag{52}$$

Solving for B_∞ in terms of p/P_e from Eqs. (51) and (52), the following approximate form of Eq. (50) is obtained:

$$\left(\frac{\sqrt{\Delta P}_e}{\xi}\right)_{max} = \frac{\frac{M_s}{M_A} - \frac{k}{(M_A/M_s)\Gamma H}}{1 - \frac{\tau}{(M_A/M_s)\Gamma H}} \tag{53}$$

This relation is plotted in Fig. 6.

Finally, the results can be put into another form which has a direct physical significance. Recasting the equations in terms of the length scale ξ of the leading-edge effect by combining Eqs. (47) and (50) and using the equation of state, one obtains

$$\frac{\xi}{\zeta} = \frac{\xi\left[p e^2 \frac{M_s}{M_A}\right]}{\left[2\pi\left(C_{h_o}\sqrt{Re}_x\right)^2\right]\left[\frac{T_{aw_o}}{T_e}M_e\right]\left[\mu_e a_e\right]} = \frac{p}{P_e}\left[\frac{\sqrt{\Delta P}_e}{\xi}\right]^2 \tag{54}$$

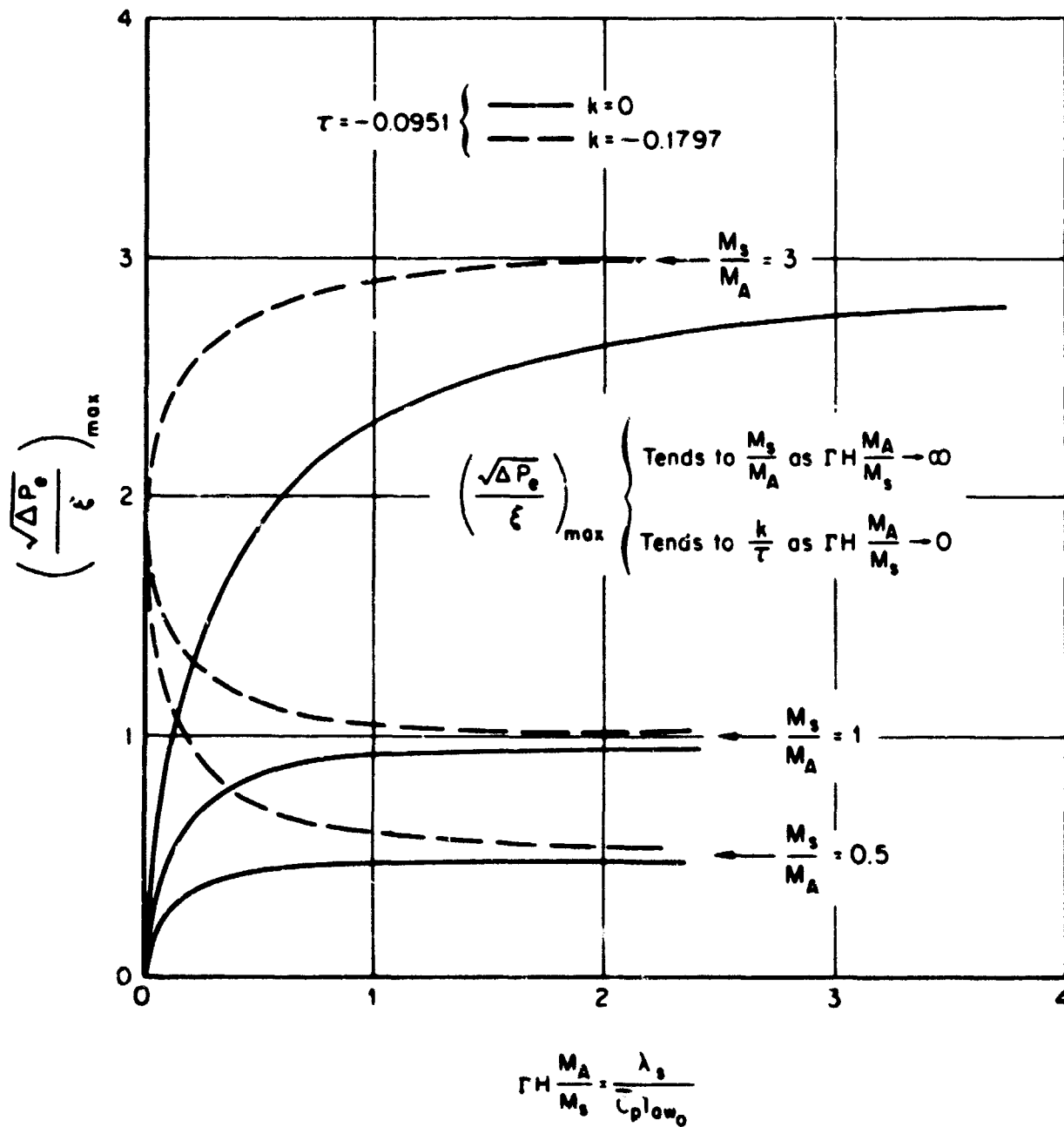$$= \frac{P_e}{p} B_\infty^2 \exp(2H)$$

-28-

Fig. 6 — Behavior of the maximum of the scale length

This is plotted in Fig. 7. Note that the three bracketed terms of the denominator on the left-hand side depend only on the basic flow field, the Mach number, and the free-stream static temperature, respectively. The bracketed term in the numerator depends only on the properties of the sublimating material. The entire factor on the left-hand side is independent of pressure. Consequently, Fig. 7 shows that during the initial re-entry from space ($p/P_e = \infty$) during which the Mach number is roughly constant (therefore H = constant and ζ = constant), δ increases to a maximum at some altitude fixed by the value of the parameter p and then decreases.
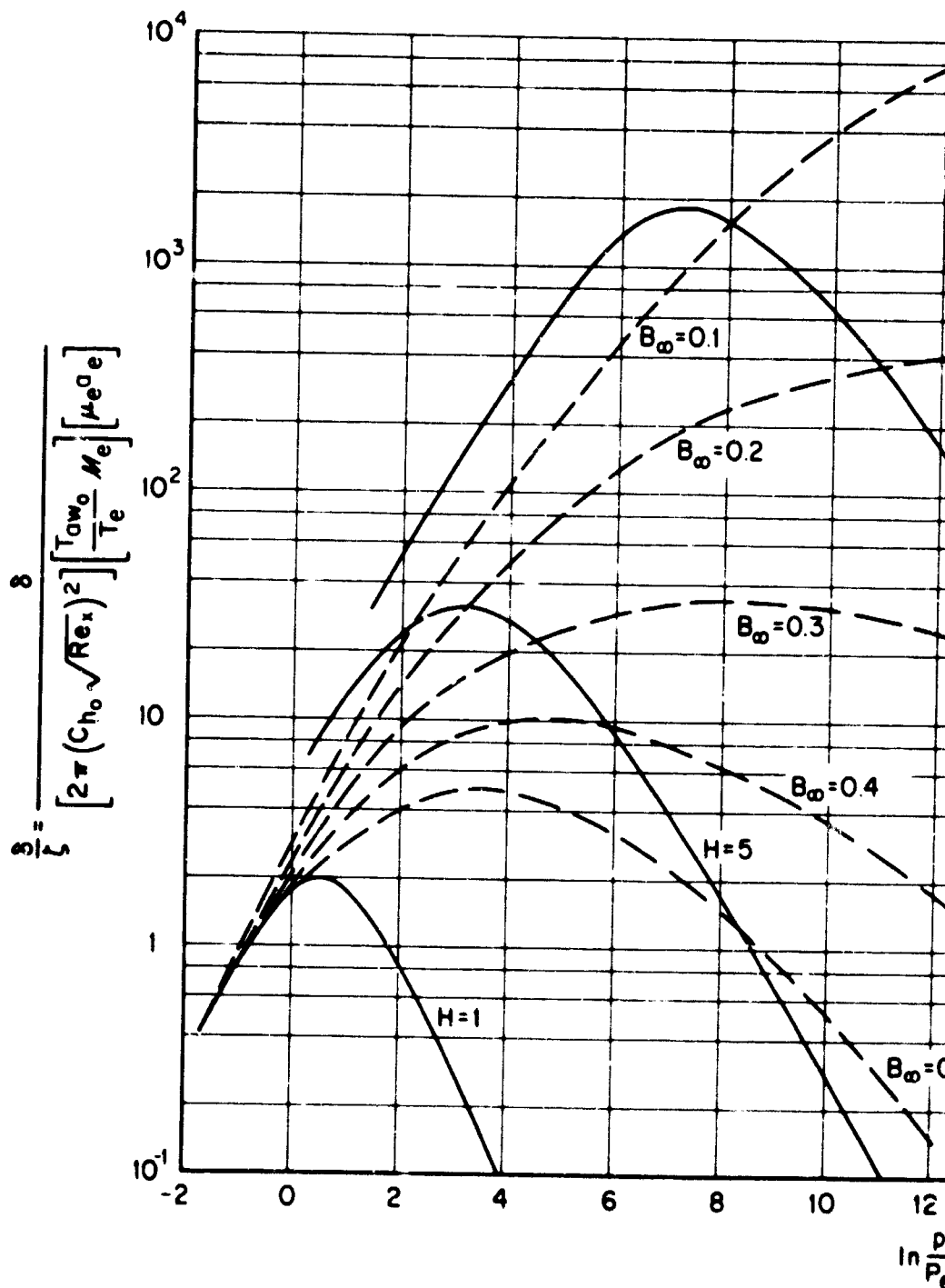
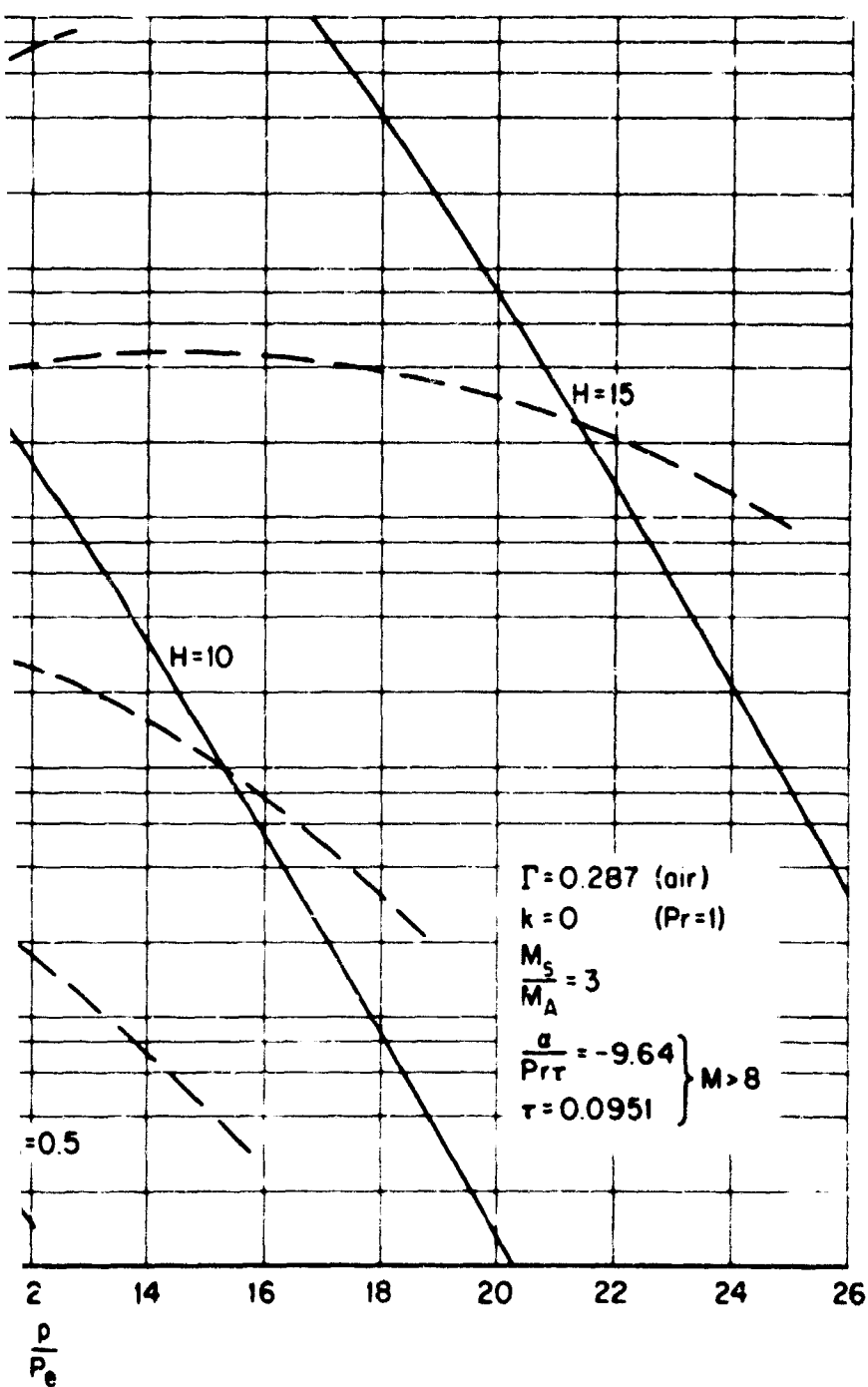$$\frac{S}{\rho_e} = \frac{S}{\left[2\pi\left(C_{h_o}\sqrt{Re_x}\right)^2\right]\left[\frac{T_{aw_o}}{T_e}\,M_e\right]\left[\mu_e a_e\right]}$$

Fig. 7 — Typical variation of the rate-controlled flow

length, showing lines of B_∞ = constant

VI. DISCUSSION

In order to show the magnitudes involved, consider the following estimates: Let the model be a slender cone for which[9]

$$c_{h_o} \sqrt{Re_x} = 0.512$$

Assume μ_e and a_e constant at their value in the tropopause ($\mu_e a_e = 4 \times 10^{-4}$ lb/ft). It follows that

$$\left[2\pi\left(c_{h_o}\sqrt{Re_x}\right)^2\right]\left[\frac{T_{aw_o}}{T_e}M_e\right]\left[\mu_e a_e\right] = 1.3 M^3 \times 10^{-4} \text{ lb/ft} \qquad (55)$$

The constant p can be obtained from vapor-pressure data. Reference 12 (pp. 1751 - 1755) gives a table for selected organic and inorganic substances from which it is evident that a representative value is $p = 10^9$ lb/ft^2.* (The value p varies around this average by one order of magnitude for almost all the substances listed.) The vaporization coefficient ϵ is poorly known. For solid sublimators it is doubtful that it should exceed 0.1,[2] and it can be less than that by two orders of magnitude. Finally, the ratio M_s/M_A can be taken as unity for the purpose of this estimate.

It follows that, quite generally,

$$10^{-11} < \frac{\epsilon}{M^3} < 10^{-9} \text{ ft} \qquad (56)$$

The magnitude of ϵ is seen to depend very strongly on the value of H. Since it increases rapidly with H, let us illustrate the problem for a large value of H: Data typical of graphite ($\epsilon = 25{,}000$ Btu/lb, $M_s = 12$) at a flight Mach number of 15 yield approximately

---

*In the notation of the reference, $p = 10^b$ where b is tabulated.

$$H = \frac{\lambda M_s}{RT_{aw_o}} \approx \frac{\lambda M_s}{RT_e} \left(1 + \frac{\gamma - 1}{2} M^2\right)^{-1} \approx 16.7$$

The value of $p = 10^9$ lb/ft^2 used in the preceding estimates represents graphite quite well--well enough considering the uncertainty in the evaporation coefficient ϵ. Since δ increases with pressure, let us consider relatively low altitudes consistent with the assumed Mach number and high-speed re-entries, say 30,000 ft. Consider a 20-deg cone for which the surface pressure is approximately 300 P_∞. It follows that

$$10^{-3} < \delta < 10^{-1} \text{ in.}$$

Had an altitude of 25,000 ft been assumed, δ would be larger by an order of magnitude. Had data typical of teflon been used ($\lambda \sim 1000$ Btu/lb, $M_s = 100$), H would have been in the order of 5, and δ would be entirely negligible.

It is interesting to note that the maximum δ for a given H occurs at values of p/P_e which correspond to P_e considerably higher than 1 atm. While pressures above atmospheric can occur, because p_e and M_e are values outside the boundary layer behind the leading shock, they would not normally be as high as indicated for δ max.

The rate-controlled region can be taken to extend over a distance from the origin equal to 100 δ; that is, for points on the surface lying beyond this limit the error incurred by using the equilibrium solution for B, $\dot{m}_s$, T_w, etc., is less than 10 per cent (see, for example, Fig. 4 or Eq. (48)). The sketch on the following page shows the behavior of the rate of sublimation indicated by both the equilibrium approximation and the full solution and illustrates the present argument.

The preceding estimates of the scale of the leading-edge effect indicate clearly that, in comparison with reasonable man-made re-entry-vehicle sizes, the region of transitional sublimation is very small, indeed normally negligible. However, δ is an independent parameter

x (distance from the leading edge)
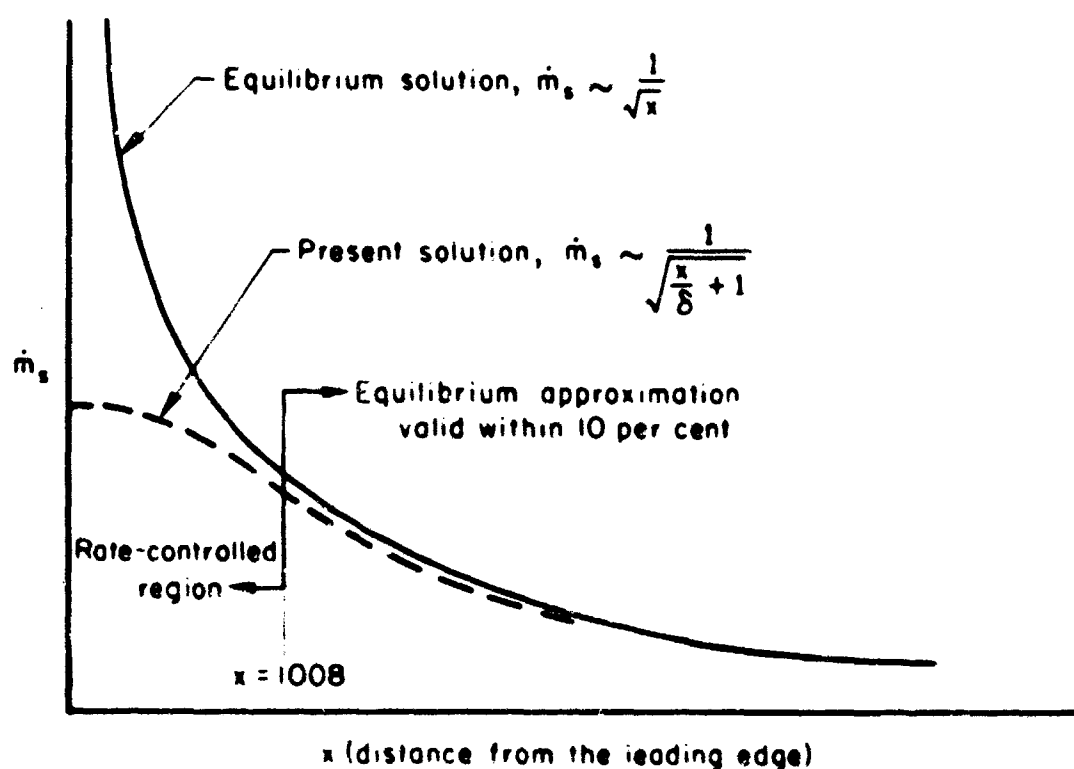
which does not scale with the body geometry. Th sublimation of small
models in wind tunnels or meteorites in the atmosphere may be totally
dominated by transitional effects.

This scaling problem can best be demonstrated by using a concrete
example. Consider the wind-tunnel experiments reported in Ref. 3 using
camphor in a Mach 5 wind tunnel. For camphor the materials data at
actual test conditions can be given quite accurately, except for ϵ:
 $\cdot$ = 330 J/gm, M_s = 152, p = 1.74 x 10^9 lb/ft^2. Estimate ϵ at 0.1 as
before.

With the tunnel recovery temperature at 350°K and the static
pressure of 10 mm Hg, one finds M = 17.2 and p/P_c = 6.3 x 10^7:

$$ \cdot ~ 2.6 \times 10^{-3} \text{ in.} $$

Considering that the transitional sublimation region extends to a dis-
tance of the order 100 (see Fig. 4), one concludes that the entire
wind-tunnel model is affected by transitional effects under these test
conditions.

One other aspect of the aerodynamics of subliming bodies can be affected by the present results in an important manner. It is the problem of surface recession and terminal shape. Briefly, the shape of a slender body $y_{(x,t)}$ at any instant t is given by the solution (with proper boundary conditions) of

$$\frac{\partial y_{(x,t)}}{\partial t} \sim \dot{m}_{s(x,t)} = \frac{K}{\sqrt{X^1} + \sqrt{\delta}} = \frac{K}{\sqrt{x - \varphi_{(t)}} + \sqrt{\delta}}$$

where K is a constant, X^1 the distance to the point (x,y) from the leading edge, which is itself receding relative to fixed coordinates at an unknown rate $\varphi_{(t)}$.

The nature of this problem is such that

$$\lim_{\delta \to 0} y_{(x,t,\ldots)} \neq y_{(x,t,\delta=0)}$$

and the analytical solution for the terminal shape of pointed bodies using the near-equilibrium form of the local sublimation rate is not correct.[*]

_____

[*] A study of this problem is in progress at The RAND Corporation.

REFERENCES

1. Gross, J. F., D. J. Masson, and C. Gazley, Jr., General Characteristics of Binary Boundary Layers with Application to Sublimation Cooling, The RAND Corporation, P-1371, Revised August 1, 1958.

2. Scala, S. M., and G. L. Vidale, "Vaporization Processes in the Hypersonic Boundary Layer," Int. J. Heat and Mass Transfer, Vol. 1, No. 1, June 1960, pp. 4-22.

3. Kubota, T., "Ablation with Ice Models at M = 5.8," ARS J., Vol. 30, No. 12, December 1960, pp. 1164-1169.

4. Scala, S. M., A Study of Hypersonic Ablation, General Electric Space Science Laboratory, Report R59SD438, September 30, 1959; presented at the 10th International Astronautical Congress, London, England, August - September 1959.

5. Knuth, E. L., "Compressible Couette Flow with Diffusion of a Reactive Gas from a Decomposing Wall," Proc. Heat Transfer and Fluid Mech. Inst., Stanford University Press, June 1958.

6. Frenkel, J., Kinetic Theory of Liquid, Clarendon Press, Oxford, 1956.

7. Lees, L., "Convective Heat Transfer with Mass Addition and Chemical Reactions," Combustion and Propulsion, Third AGARD Colloquium, Palermo, Sicily, March 1958.

8. Li, T. Y., and J. F. Gross, Hypersonic Strong Viscous Interaction on a Flat Plate with Surface Mass Transfer, The RAND Corporation, RM-3000-PR (Abridged), March 1963; also Proc. 1961 Heat Transfer and Fluid Mech. Inst., Stanford University Press, 1961.

9. Low, G. M., The Compressible Laminar Boundary Layer with Fluid Injection, NACA TN 3404, 1955.

10. Faulders, C. R., "Heat Transfer in the Laminar Boundary Layer with Ablation of Vapor of Arbitrary Molecular Weight," J. Aerospace Sci., Vol. 29, No. 1, January 1962, pp. 76-86.

11. Li, T. Y., Similar Solutions of Compressible Laminar-Boundary-Layer Equations for Binary Mixtures, The RAND Corporation, RM-2523, March 9, 1960.

12. Handbook of Chemistry and Physics, 20th ed., Chemical Rubber Publishing Company, Cleveland, Ohio, 1935.